MATH 245 F17, Exam 3 Solutions

1. Carefully define the following terms: recurrence, Ω, Δ, $=$ (for sets).

A recurrence is a sequence, such that all but finitely many terms are defined in terms of its previous terms. Given two sequences a_n, b_n, we say that $a_n = \Omega(b_n)$ to mean that there is some $n_0 \in \mathbb{N}$ and there is some $M \in \mathbb{R}$ such that $\forall n \geq n_0$, $M|a_n| \geq |b_n|$. Given sets R, S, the set $R\Delta S = \{x : (x \in R \wedge x \notin S) \vee (x \notin R \wedge x \in S)\}$. Given sets R, S, we say that $R = S$ if they contain the exact same elements.

2. Carefully define the following terms: disjoint, equicardinal, Distributivity Theorem (for sets), De Morgan's Law Theorem (for sets).

Given sets R, S, we say they are disjoint if $R \cap S = \emptyset$. Given sets R, S, we say they are equicardinal if we can pair the elements of R with the elements of S. Given sets R, S, T, the Distributivity Theorem states $R \cap (S \cup T) = (R \cap S) \cup (R \cap T)$ and $R \cup (S \cap T) = (R \cup S) \cap (R \cup T)$. Given sets R, S, U with $R \subseteq U$ and $S \subseteq U$, De Morgan's Law states that $(R \cap S)^c = R^c \cup S^c$ and $(R \cup S)^c = R^c \cap S^c$.

3. Let $S = \{a, b\}$. Give a two-element subset of 2^{2^S}.

We seek a set, both elements of which are elements of 2^{2^S}. That is, we need a set, both elements of which are subsets of $2^S = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$. Many solutions are possible, such as $\{\{\emptyset, \{a\}\}, \{\emptyset, \{b\}\}\}$, or $\{\emptyset, \{\emptyset\}\}$, or $\{\{\{a\}\}, \{\{b\}\}\}$. Careful notation is critical here.

4. Suppose that a recurrence satisfies the relation $T_n = 4T_{n/2} + n^2$. Determine what, if anything, the Master Theorem tells us.

We have $a = 4, b = 2$, and $d = \log_2 4 = 2$. Because $c_n = n^2 = n^d$, in fact $c_n = \Theta(n^d)$. Hence, the "Middle c_n" part of the theorem applies, which tells us that $T_n = \Theta(n^2 \log n)$.

5. Let R, S, T be sets, with $S \subseteq T$. Prove that $R \cap S \subseteq R \cap T$.

Let $x \in R \cap S$. Hence $x \in R \wedge x \in S$. By simplification, $x \in S$. Because $S \subseteq T$, in fact $x \in T$. By simplification on $x \in R \wedge x \in S$ the other way, $x \in R$. Applying conjunction to $x \in R$ and $x \in T$, we get $x \in R \wedge x \in T$. Hence $x \in R \cap T$.

6. Let R, S, U be sets, with $R \subseteq S \subseteq U$. Prove that $S^c \subseteq R^c$.

Let $x \in S^c$. Hence $x \in U \setminus S$ and thus $x \in U \wedge x \notin S$. By simplification twice, we get $x \in U$ and $x \notin S$. We now have two cases, depending on whether or not $x \in R$: If $x \in R$, then (since $R \subseteq S$), $x \in S$. But this is impossible, so this case can't happen. If instead $x \notin R$, then, by conjunction, $x \in U \wedge x \notin R$. Hence $x \in U \setminus R$ and so $x \in R^c$.

7. Prove or disprove: For all sets R, S, $R \times S = S \times R$.

The statement is false, and needs a counterexample to disprove. This will be specific sets R, S to falsify the equality. Many solutions are possible, such as $R = \{1\}, S = \{2, 3\}$. Now $R \times S = \{(1, 2), (1, 3)\}$ while $S \times R = \{(2, 1), (3, 1)\}$. To falsify the equality we need a specific element of one set, that is not an element of the other. Note that $(1, 2) \in R \times S$ but $(1, 2) \notin S \times R$, so $R \times S \neq S \times R$.

8. Solve the recurrence given by $a_0 = a_1 = 1, a_n = 5a_{n-1} - 6a_{n-2}$ $(n \geq 2)$.

The characteristic polynomial is $r^2 = 5r - 6$, which rearranges as $0 = r^2 - 5r + 6 = (r - 3)(r - 2)$. Hence the general solution is $a_n = A3^n + B2^n$. We now use our initial conditions as $1 = a_0 = A3^0 + B2^0 = A + B$, and $1 = a_1 = A3^1 + B2^1 = 3A + 2B$. This has solution $A = -1, B = 2$, so our solution is $a_n = -3^n + 2 \cdot 2^n = 2^{n+1} - 3^n$.

9. Let $a_n = 3n^2 + 7$. Prove that $a_n = \Theta(n^2)$.

Part 1 ($n^2 = O(a_n)$): Take $n_0 = M = 1$, and let $n \geq n_0 = 1$. We have $|n^2| = n^2 \leq 3n^2 + 7 = M|3n^2 + 7| = M|a_n|$.
Part 2 ($a_n = O(n^2)$): Take $n_0 = 7, M = 4$, and let $n \geq n_0 = 7$. We have $n^2 \geq 7n \geq 7$, so $|a_n| = |3n^2 + 7| = 3n^2 + 7 \leq 3n^2 + n^2 = 4n^2 = M|n^2|$.

10. Let R, S, T be sets. Prove that $R \times (S \cup T) \subseteq (R \times S) \cup (R \times T)$.

Let $x \in R \times (S \cup T)$. Then $x = (a, b)$ where $a \in R$ and $b \in S \cup T$. We have two cases. Case 1: $b \in S$. Then, $(a, b) \in R \times S$, so $x \in R \times S$. By addition, $x \in R \times S \vee x \in R \times T$, so $x \in (R \times S) \cup (R \times T)$. Case 2: $b \in T$. Then, $(a, b) \in R \times T$, so $x \in R \times T$. By addition, $x \in R \times S \vee x \in R \times T$, so $x \in (R \times S) \cup (R \times T)$. In either case, $x \in (R \times S) \cup (R \times T)$.